

A LOWER BOUND FOR TOPOLOGICAL ENTROPY OF GENERIC NON ANOSOV SYMPLECTIC DIFFEOMORPHISMS

THIAGO CATALAN AND ALI TAHZIBI

ABSTRACT. We prove that a C^1 -generic symplectic diffeomorphism is either Anosov or the topological entropy is bounded from below by the supremum over the smallest positive Lyapunov exponent of the periodic points. We also prove that C^1 -generic symplectic diffeomorphisms outside the Anosov ones do not admit symbolic extension and finally we give examples of volume preserving diffeomorphisms which are not point of upper semicontinuity of entropy function in C^1 -topology.

1. INTRODUCTION

The topological entropy is one of the most important topological invariants for dynamical systems. Informally, the topological entropy calculates the “number of different trajectories” of the dynamics. Formally, we define it in the following way

$$h(f) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r(n, \varepsilon);$$

where $r(n, \varepsilon)$ is the maximum amount of ε -distinct orbits of length n . Two points have ε -distinct orbits of length n if there is $0 \leq j \leq n$ such that $d(f^j(x), f^j(y)) > \varepsilon$.

For Axiom A diffeomorphisms, Bowen [6] proved that entropy determines the asymptotical exponential growth of the number of periodic points and by a Katok’s result for any $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a two dimensional manifold entropy is bounded above by such growth rate: $h(f) \leq \limsup_{n \rightarrow \infty} \frac{P_n(f)}{n}$.

In this paper we prove lower estimates for topological entropy of C^1 -generic symplectic dynamics in terms of Lyapunov exponents of periodic points of the system (See theorems B, A).

We relate such lower bounds for the entropy to the (semi continuity) regularity of the entropy function with respect to the dynamics (see Theorem C) and construct examples of surface diffeomorphisms which are not point of semi continuity of the topological entropy.

Date: November 11, 2010.

2000 AMS Subject Classification: 37C10 (Primary), 37D25 (Secondary). *Key words and phrases:* Topological Entropy, Symbolic Extensions, Homoclinic Tangency, Lyapunov Exponents. Partially supported by CAPES and FAPESP.

Finally, we take benefit of the estimates in order to prove the non existence of symbolic extensions for C^1 -generic symplectic diffeomorphisms far from the Anosov ones (See theorem D).

1.1. Lower estimates for topological entropy. Firstly, Newhouse in [17] got some lower bound for topological entropy of generic area preserving diffeomorphisms on surfaces far from Anosov diffeomorphisms. Remember for a C^1 diffeomorphism over some manifold M be Anosov means that the whole manifold M is a hyperbolic set for f , where an f -invariant compact set Λ of M is a *hyperbolic set* if there is a continuous and Df -invariant splitting $T_\Lambda M = E^s \oplus E^u$ such that there are constants $0 < \lambda < 1$ and $C > 0$, satisfying

$$\|Df_x^k|E^s(x)\| \leq C\lambda^k \quad \text{and} \quad \|Df_x^{-k}|E^u(x)\| \leq C\lambda^k,$$

for every $x \in \Lambda$ and $k > 0$.

More precisely, let M be a compact, connected surface with a volume form m , and denote by $\text{Diff}_m^1(M)$ the set of conservative C^1 diffeomorphisms, i.e., the set formed by diffeomorphisms that preserve the volume form m . Taking

$$s(f) = \sup \left\{ \frac{1}{\tau(p, f)} \log \lambda(p, f) \right\}$$

over all hyperbolic periodic points p of f , where $\tau(p, f)$ is the minimum period of the hyperbolic periodic point p , and $\lambda(p, f)$ is the absolute value of the unique eigenvalue of $Df^{\tau(p, f)}(p)$ with absolute value larger than one, Newhouse's result is the following.

1.1. Theorem. (Newhouse) *There exists a residual subset $\mathcal{B} \subset \text{Diff}_m^1(M)$ such that if $f \in \mathcal{B}$ is a non Anosov diffeomorphism then*

$$h(f) \geq s(f).$$

Here we show that indeed generically the reverse inequality also holds and this implies:

Theorem A. *There exists a residual subset $\mathcal{B} \subset \text{Diff}_m^1(M)$ (volume preserving surface diffeomorphisms) such that if $f \in \mathcal{B}$ is a non Anosov diffeomorphism then*

$$h(f) = s(f).$$

Observe as a corollary of this theorem and semi continuity of $f \rightarrow s(f)$ (see preliminary definitions in the next section) we conclude that “generically” topological entropy is semi continuous in C^1 -topology. However, it is not known whether the semi-continuity points of topological entropy form a C^1 -generic subset.

It is interesting to mention that among Anosov diffeomorphisms, diffeomorphisms of a C^1 -open and dense subset satisfy $h(f) < s(f)$. See Proposition 2.1 which gives an upper bound for the entropy of Anosov diffeomorphisms.

In general it may exist diffeomorphisms where $h(f) > s(f)$. In fact there exist even minimal diffeomorphism with positive entropy. In two dimensional case there are minimal homeomorphisms with positive entropy [20]. However, these examples are not volume preserving (and should be outside a C^1 -generic subset).

1.2. Question. *Is there an example of a conservative C^1 surface diffeomorphism where $h(f) > s(f)$?*

Our next result is a generalization of Newhouse's theorem to higher dimensional symplectic setting. Let (M, ω) be a compact, connected, smooth Riemannian symplectic manifold. For a hyperbolic periodic point p of f , we denote by $\lambda(p, f)$ the absolute value of the smallest eigenvalue of $Df^{\tau(p, f)}(p)$ between those ones with absolute value bigger than one. Define

$$s(f) := \sup \left\{ \frac{1}{\tau(p, f)} \log \lambda(p, f) \right\}$$

over all hyperbolic periodic points p of f , and then what we prove is the following.

Theorem B. *There exists a residual subset $\mathcal{B} \subset \text{Diff}_\omega^1(M)$ such that if $f \in \mathcal{B}$ is a non Anosov diffeomorphism then*

$$h(f) \geq s(f).$$

1.3. Question. *What about Conservative case? What can be said if we define $s(f)$ as the supremum over the sum of all positive Lyapunov exponents of periodic points instead of just the smallest Lyapunov exponent of periodic points?*

1.2. Regularity of Entropy. An important problem in smooth ergodic theory is the regularity of entropy with respect to dynamics. By Newhouse result we know that $f \rightarrow h(f)$ is upper semicontinuous in the C^∞ topology for any compact boundaryless manifold and using Katok's result it is indeed continuous for C^∞ surface diffeomorphisms. Here we show that

Theorem C. *There are examples of surface diffeomorphisms $f_0 \in \text{Diff}^\infty(M)$ such that $f \rightarrow h(f)$ is not even upper semi continuous in the C^1 -topology at f_0 .*

1.3. Symbolic Extensions. Symbolic dynamics play a crucial role in ergodic theory. It is a challenging problem to know whether a dynamics can be codified. We can obtain upper bounds for the entropy by means of symbolic dynamics: A dynamical systems (M, f) have a *symbolic extension* if there exist a subshift (Y, σ) and a surjective map $\pi : Y \rightarrow M$ such that $\pi \circ \sigma = f \circ \pi$. (Y, σ) is called an *extension* of (M, f) and (M, f) a *factor* of (Y, σ) . And so, if the system has some

symbolic extension we gain directly an upper bound of the topological entropy. Nevertheless, such estimate could be extreme. This way, one extension is called *principal extension* if the map π is such that $h_\nu(\sigma) = h_{\pi_*\nu}(f)$ for every σ -invariant measure $\nu \in \mathcal{M}(\sigma)$ over Y , where $h_\nu(\sigma)$ is the metric entropy of σ with respect to ν .

Boyle, D. Fiebig, U. Fiebig [7] proved that asymptotically h -expansive diffeomorphisms have a principal symbolic extension. By a result of Buzzi [9] every C^∞ diffeomorphism of compact manifold is asymptotically entropy expansive and consequently have a principal symbolic extension. Also, recently D. Burguet [8] showed that every C^2 surface diffeomorphism have symbolic extensions. These results give a positive partial answer to the conjecture of Downarowicz and Newhouse which expects symbolic extension for any C^r ($r \geq 2$) diffeomorphism.

Let us mention that Diaz, Fisher, Pacífico and Vieitez [11] proved that every C^1 partially hyperbolic diffeomorphism with a nonhyperbolic central bundle that splits in a dominated way into 1-dimensional sub-bundles is asymptotically h -expansive and therefore has a principal symbolic extension. See also [10].

On the other hand, Downarowicz and Newhouse using Theorem 1.1 proved in [16] that far from Anosov diffeomorphisms, generic area preserving diffeomorphisms in C^1 topology admit no symbolic extensions. We extend this result to higher dimensional symplectic diffeomorphisms.

Theorem D. *There is a residual subset $\mathcal{B} \subset \text{Diff}_\omega^1(M)$ such that if $f \in \mathcal{B}$ is a non Anosov diffeomorphism then f has no symbolic extension.*

Now, using this result we be able to give an easy and short prove of the stability conjecture in symplectic scenario. We say that a symplectic diffeomorphism is structurally stable if there is some neighborhood \mathcal{U} of f in $\text{Diff}_\omega^1(M)$ such that every diffeomorphism $g \in \mathcal{U}$ is topologically conjugated to f , i.e., there is one homeomorphism h over M such that $hf = gh$.

1.4. Corollary. *A diffeomorphism $f \in \text{Diff}_\omega^1(M)$ is structurally stable if, and only if, f is Anosov.*

Proof. Suppose $f \in \text{Diff}_\omega^1(M)$ is a structurally stable diffeomorphism. Now, by Zehnder [24], smooth diffeomorphisms are dense among the symplectic ones and since C^∞ diffeomorphisms have a principal symbolic extension, every diffeomorphism in some neighborhood of f also has a principal symbolic extension. So, accordingly to Theorem D this is only possible if f is Anosov. \square

1.5. Remark. *The proof of the above corollary is based on Theorem D, but to prove it we use the unfolding of homoclinic tangency outside Anosov set that happens in the symplectic scenario, which is by itself an obstruction of stability. Nevertheless, we would like to emphasize that a priori non existence of symbolic extensions does not*

have direct relation to homoclinic tangencies. So, we conclude that “any mechanism in C^1 -topology” which yields non existence of symbolic extensions implies non structural stability.

However, very recently we were informed that G.Liao, J.Yang and M. Viana [15] proved that diffeomorphisms C^1 -far from tangencies have symbolic extensions.

This paper is organized as follows. In section 2 we will prove some generic results relating topological entropy with Lyapunov exponents of periodic points, in particular, we prove Theorem A. In section 3, we prove Theorems B and C using a main technical proposition (Proposition 3.1). In section 4, the main technical proposition is proved and finally in section 5 we give examples of (upper semi) discontinuity points for topological entropy.

2. ENTROPY AND LYAPUNOV EXPONENTS OF PERIODIC POINTS

In this section firstly we review some background definitions and results, and prove an strict upper bound for the entropy in a C^1 -open and dense subset of Anosov volume preserving diffeomorphisms. After, using results of Abdenur, Bonatti and Crovisier [1] we prove an upper bound for entropy of C^1 -generic volume preserving diffeomorphisms of compact manifolds (any dimension) and apply it to prove Theorem A.

2.1. Preliminary definitions. Given $f \in \text{Diff}_\omega^1(M)$ and a hyperbolic periodic point p of f , we denote by $\chi(p, f)$ the smallest positive Lyapunov exponent for the hyperbolic periodic point p of f , i.e., $\chi(p, f) = 1/\tau(p, f) \log \lambda(p, f)$, where $\lambda(p, f) = (\sigma(Df^{-\tau(p, f)}|E^u))^{-1}$, being σ the spectral radio of the map, and as before $\tau(p, f)$ is the minimum period of the hyperbolic periodic point p . In fact, defining

$$\mu_p = \frac{1}{\tau(p, f)} \sum_{i=0}^{\tau(p)-1} \delta_{f^i(p)},$$

as the *periodic measure* for p , where $\delta_{f^i(p)}$ is the dirac measure for $f^i(p)$, we have that $\chi(p, f)$ is the smallest positive lyapunov exponent for the ergodic measure μ_p .

Now, given $n \in \mathbb{N}$ we consider $s_n(f) = \max \{\chi(p, f); p \in H_n(f)\}$, where $H_n(f)$ is the set of hyperbolic periodic points of period smaller or equal than n . Since $H_n(f) \subset H_{n+1}(f)$, we have $s_n(f) \leq s_{n+1}(f)$, and then it's well defined $s(f) = \lim_{n \rightarrow \infty} s_n(f)$. From robustness of the hyperbolic periodic points we have that the functional s_n is continuous for every $n \in \mathbb{N}$, which implies that $s(f)$ is lower semicontinuous.

2.2. Generic upper bound for entropy of Anosov diffeomorphisms. In the setting of volume preserving Anosov diffeomorphisms the scenario is much more clear.

2.1. Proposition. *There exists a C^1 -open and dense subset \mathcal{F} of volume preserving Anosov diffeomorphisms such that for any $f \in \mathcal{F}$ with $\dim(E^u) = u$ we have*

$$h(f) < \sup_{p \in \text{Per}(f)} \sum_{i=1}^u \chi_i^+(p)$$

where the sum is over all positive Lyapunov exponents of p .

Proof. Take any volume preserving Anosov diffeomorphism f . After a small C^1 -perturbation, if necessary, we can assume that f is a C^2 Anosov diffeomorphism. This regularization result is due to Avila [5]. Now, we know that for volume preserving Anosov diffeomorphisms the Lebesgue measure m is the unique ergodic equilibrium state for the potencial $\phi^u(\cdot) = -\log J^u(f)$ where $J^u(f) := |\det Df|E^u(f)|$. Recall that the entropy maximizing measure is just the equilibrium state for the identically zero potential. Using Bowen's result it is clear that μ , the entropy maximizing measure, coincides with Lebesgue if, and only if, the potential ϕ^u is cohomologous to a constant function. So, perturbing f in C^1 -topology we can assume that μ is singular with respect to the Lebesgue measure.

Following, recall that in Bowen's approach the entropy maximizing measures are obtained as the limit of periodic distributions. That is

$$\mu_n := \frac{\sum_{p \in \text{Per}_n(f)} \delta_p}{\#\text{Per}_n(f)} \rightarrow \mu$$

Since $\phi^u(\cdot)$ is a continuous function we have $\int -\phi^u(x) d\mu(x) = \lim_{n \rightarrow \infty} \int -\phi^u(x) d\mu_n(x)$ and by definition we conclude that $\int -\phi^u d\mu_n \leq \sup_{p \in \text{Per}_n(f)} \sum_{i=1}^u \chi_i^+(p)$. Then,

$$\int -\phi^u d\mu \leq \sup_{p \in \text{Per}(f)} \sum_{i=1}^u \chi_i^+(p).$$

Now, provided f is Anosov the pressure of $\phi^u(\cdot)$ is zero. Hence,

$$\begin{aligned} 0 = P_f(\phi^u) &= h_m(f) + \int \phi^u dm \\ &> h_\mu(f) + \int \phi^u d\mu \\ &= h(f) + \int \phi^u d\mu. \end{aligned}$$

So it came out that

$$h(f) < \int -\phi^u d\mu \leq \sup_{p \in \text{Per}(f)} \sum_{i=1}^u \chi_i^+(p).$$

Finally, we claim that any C^1 -perturbation of f also satisfies a similar inequality. Indeed, as entropy is locally constant (by structural stability of Anosov

diffeomorphisms) and $f \rightarrow \sup_{p \in \text{Per}(f)} \sum_{i=1}^u \chi_i^+(p)$ is a lower semi continuous function we conclude that for any g C^1 -close enough to f we have

$$h(g) < \sup_{p \in \text{Per}(g)} \sum_{i=1}^u \chi_i^+(p).$$

□

2.3. Generic upper bounds for entropy and proof of Theorem A. Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold and μ an ergodic hyperbolic measure. Then Katok proved, see [13], that there exists a sequence of periodic points p_n such that dirac measures on the orbit of p_n converge to μ and the Lyapunov exponents of p_n converge to the Lyapunov exponent of μ . By variational principle $h(f) = \sup_{\mu} h_{\mu}(f)$ where the supremum is over all f -invariant ergodic probability measures. By Ruelle's inequality, $h_{\mu}(f) \leq \sum \chi_i^+$ where the sum is over all positive Lyapunov exponents of μ . Suppose that the supremum in the variational principle can be taken over hyperbolic measures. Then we conclude that

$$h(f) \leq \sup_{p \in \text{Per}(f)} \sum \chi_i^+(p)$$

Using Abdenur, Bonatti and Crovisier's ideas we show it is the case for C^1 -generic volume preserving diffeomorphisms.

Theorem A will be a consequence of Theorem 1.1 and the following theorem.

2.2. Theorem. *There exists a residual subset $\mathcal{R} \subset \text{Diff}_m^1(M)$ (M of any dimension d) such that for any $f \in \mathcal{R}$*

$$h(f) \leq \sup_{p \in \text{Per}(f)} \sum_{i=1}^{n_p} \chi_i^+(p)$$

where the sum is over all positive Lyapunov exponents of the periodic point p , counting multiplicity.

Let us see first, how to prove Theorem A. Take \mathcal{R} as the residual subset derived from the intersection of the ones given by Theorem 1.1 and Theorem 2.2. If the supremum in Theorem 2.2 was taken over hyperbolic periodic points then this sup in dimension two would be equal $s(f)$ and then Theorem A was proved. In order to overcome this, we divide the proof in two cases. Given a diffeomorphism $f \in \mathcal{R}$, if $h(f) = 0$ then we have equality by Theorem 1.1 ($s(f) \geq 0$), i.e., $h(f) = s(f)$. On the other hand, since in dimension two for a periodic point of a conservative diffeomorphism be hyperbolic it's enough to have positive lyapunov exponent, if $h(f) > 0$ we have that in Theorem 2.2 the supremum is in fact over the hyperbolic periodic points, and then, we also have equality between $h(f)$ and $s(f)$, which concludes Theorem A.

In the sequence we prove Theorem 2.2.

Using Abdenur, Bonatti and Crovisier [1], we can prove the following Proposition.

2.3. Proposition. *There is a residual subset $\mathcal{R} \subset \text{Diff}_m^1(M)$ such that if $f \in \mathcal{R}$ and μ is an ergodic measure for f , then there are periodic measures μ_p converging to μ in the weak topology, and moreover the vectors formed by the lyapunov exponentes of μ_p , $L(\mu_p) \in \mathbb{R}^d$, also converge to the Lyapunov vector $L(\mu) \in \mathbb{R}^d$.*

In fact, they proved this result for dissipative diffeomorphisms, Theorem 3.8 in [1]. But, unless generic arguments which are also true in the conservative setting, their theorem is a consequence of Proposition 6.1 there, which we state here for simplicity.

2.4. Proposition. *Let μ be an ergodic invariant probability measure of a diffeomorphism f of a compact manifold M . Fix a C^1 -neighborhood \mathcal{U} of f , a neighborhood \mathcal{V} of μ in the space of probability measures with the weak topology, a Hausdorff neighborhood \mathcal{K} of the support of μ , and a neighborhood \mathcal{O} of $L(\mu)$ in \mathbb{R}^d . Then there is $g \in \mathcal{U}$ and a periodic point p of g such that the Dirac measure μ_p associated to p belongs to \mathcal{V} , its support belongs to \mathcal{K} , and its Lyapunov vector $L(\mu_p)$ belongs to \mathcal{O} .*

They divided the proof of this proposition in two lemmas, Lemma 6.2 and Lemma 6.3 there. In the first one, given an ergodic measure μ for f they found strategic periodic points p_n of some diffeomorphisms $f_n \in \mathcal{U}$ with good properties. Then, in Lemma 6.3 they proved that in fact the Lyapunov vectors $L(\mu_{p_n})$ converge to $L(\mu)$.

Hence, since we want this result in the conservative world, given some conservative diffeomorphism f we need to find conservative diffeomorphisms f_n in any neighborhood of f with same good properties as in their Lemma 6.2. Fortunately, since we have the ergodic closing lemma, see [4], and Frank's lemma, see [14] and [2], in the conservative scenario, the proof of a conservative version of Proposition 2.4 is exactly the same.

Now, using Proposition 2.3 we prove Theorem 2.2.

Proof of Theorem 2.2: Let $f \in \mathcal{R}$, where \mathcal{R} is the residual set as in Proposition 2.3. Given any $\varepsilon > 0$, by variational principle there is an ergodic measure $\mu \in \mathcal{M}(f)$ such that

$$h(f) < h_\mu(f) + \varepsilon.$$

By Ruelle's inequality $h_\mu(f) \leq \sum \chi_i^+(\mu)$, where the sum is over all positive Lyapunov exponentes of μ . Now, by Proposition 2.3 there is a periodic point p of f such that $\sum \chi_i^+(\mu) < \sum \chi_i^+(\mu_p) + \varepsilon$. And then, we have

$$h(f) < \sup_{p \in \text{Per}(f)} \sum_i^+ \chi_i(p) + 2\varepsilon.$$

Therefore, since ε is arbitrarily small we prove the theorem. \square

3. ENTROPY ESTIMATE FOR SYMPLECTOMORPHISMS

If f is a C^1 diffeomorphism over some manifold M , and p is a hyperbolic periodic point of f , we denote by $H(p, f)$ the set of transversal homoclinic points of p , where $q \notin o(p)$ is a *transversal homoclinic point* if is a transversal intersection point of $W^s(o(p), f)$ and $W^u(o(p), f)$. If the intersection is not transversal we say that q is a point of *homoclinic tangency*.

Zhihong Xia in [22] proved that there exists a residual subset $\mathcal{H} \subset \text{Diff}_\omega^1(M)$ such that if $f \in \mathcal{H}$ and p is a hyperbolic periodic point of f then transversal homoclinic points are dense on stable and unstable manifolds, $W^s(o(p), f) \cup W^u(o(p), f) \subset \text{cl}(H(p, f))$. Let us now state the main proposition and prove Theorems B and D. We postpone the proof of this proposition to the next section. Recall $\chi(p, f) = 1/\tau(p, f) \log \lambda(p, f)$ is the smallest positive lyapunov exponent for a hyperbolic periodic point p of f .

3.1. Proposition. (Main Technical Proposition) *Let p be a hyperbolic periodic point of some non Anosov diffeomorphism $f \in \mathcal{H}$. Given $n > 0$ and any neighborhood $\mathcal{N} \subset \text{Diff}_\omega^1(M)$ of f , there exists an open set $\mathcal{U} \subset \mathcal{N}$ such that if $g \in \mathcal{U}$, then g has a basic hyperbolic set $\Lambda(p(g), n) \subset \text{cl}(H(p(g), g))$, where $p(g)$ is the continuation of the hyperbolic periodic point p of f for g , such that the following properties are true*

- a) $h(g|\Lambda(p(g), n)) > \chi(p(g), g) - \frac{1}{n}$.
- b) *There exists an ergodic measure $\mu \in \mathcal{M}(\Lambda(p(g), n))$ such that*

$$h_\mu(g) > \chi(p(g), g) - \frac{1}{n}.$$

- c) *For every ergodic measure $\mu \in \mathcal{M}(\Lambda(p(g), n))$, we have*

$$\rho(\mu, \mu_{p(g)}) < \frac{1}{n},$$

where ρ is a metric which generates the weak topology.

- d) *For every periodic point $q \in \Lambda(p(g), n)$, we have*

$$\chi(q, g) > \chi(p(g), g) - \frac{1}{n}.$$

3.1. Proof of Theorem B. We denote by \mathcal{A} the set of Anosov diffeomorphisms and consider $\mathcal{D} = \text{Diff}_\omega^1(M) - \text{cl}(\mathcal{A})$, the complement of the closure of Anosov diffeomorphisms.

For positive integers n and m , let $B_{n,m}$ be the set of diffeomorphisms f in \mathcal{D} such that there are $p \in H_n(f)$ and a hyperbolic basic set $\Lambda \subset \text{cl}(H(p, f))$, satisfying

$$h(f|\Lambda) > s_n(f) - \frac{1}{m}.$$

Theorem B follows immediately from the next claim.

Claim: $B_{n,m}$ is an open and dense subset of \mathcal{D} , for every positive integers n and m .

To prove the claim, we can start with $f \in \mathcal{D} \cap \mathcal{H}$ since we are in a Baire space. Now, let $n, m \in \mathbb{N}$ be anyone.

By definition of s_n , there exists $p_0 \in H_n(f)$ such that

$$s_n(f) = \chi(p_0, f).$$

Using Proposition 3.1, we can find f_1 C^1 -close to f such that f_1 has a hyperbolic basic set $\Lambda \subset cl(H(p_0(f_1), f_1))$, where $p_0(f_1)$ is the continuation of p_0 for f_1 , and

$$(1) \quad h(f_1|\Lambda) > \chi(p_0(f_1), f_1) - \frac{1}{3m}.$$

Now, using the robustness of Λ and p_0 , the invariance of topological entropy and that s_n is continuous, we have the following for g C^1 -near f_1

$$\begin{aligned} h(g) &\geq h(g|\Lambda(g)) \\ &= h(f_1|\Lambda) \\ &> \chi(p_0(f_1), f_1) - \frac{1}{3m} \\ &\geq \chi(p_0, f) - \frac{2}{3m} \\ &= s_n(f) - \frac{2}{3m} \\ &> s_n(g) - \frac{1}{m}, \end{aligned}$$

which proves the claim and then Theorem B. □

3.2. Proof of Theorem D. Remember that (Y, σ) is a symbolic extension of (M, f) if there exists a continuous surjective map $\pi : Y \rightarrow M$ such that $\pi \circ \sigma = f \circ \pi$. As we already comment, it may happen that symbolic extensions of a system have larger entropy and carry much more information than the system.

Hence, let

$$h_{ext}^\pi(\mu) = \sup\{h_\nu(\sigma|Y) : \pi_*\nu = \mu\}, \text{ for } \mu \in \mathcal{M}(f),$$

and observe that principal symbolic extensions minimize these functions.

Let $S(f)$ be the set of all possible symbolic extensions (Y, σ, π) of (M, f) . We say that $S(f) = \emptyset$ if there is no symbolic extension of (M, f) . We define the *residual entropy of the system* by

$$h_{res}(f) = h_{sex}(f) - h(f),$$

where

$$h_{sex}(f) = \begin{cases} \inf\{h_{ext}^\pi(\mu) : (Y, \sigma, \pi) \in S(f)\} & \text{if } S(f) \neq \emptyset \\ \infty & \text{if } S(f) = \emptyset \end{cases}$$

In terms of this, to prove Theorem D we need to show that $h_{\text{sex}}(f) = \infty$ for all non Anosov diffeomorphism f in some residual subset $\mathcal{B} \subset \text{Diff}_\omega^1(M)$.

Given $f : M \rightarrow M$ a homeomorphism in a compact metric space M , an increasing sequence $\alpha_1 \leq \alpha_2 \leq \dots$ of partitions of M is called *essential* for f if

1. $\text{diam}(\alpha_k) \rightarrow 0$ when $k \rightarrow \infty$, and
2. $\mu(\partial\alpha_k) = 0$ for every $\mu \in \mathcal{M}(f)$. Where $\partial\alpha_k$ denotes the union of boundaries of all elements of the partition α_k .

A *sequence of simplicial partitions* is a nested sequence $\mathcal{S} = \{\alpha_1, \alpha_2, \dots\}$ of partitions whose diameters go to zero, and each α_k is given by some smooth triangulation of M . By Proposition 4.1 in [16] there is a residual subset $\mathcal{R}_\mathcal{S} \subset \text{Diff}_\omega^1(M)$ such that if $f \in \mathcal{R}_\mathcal{S}$ then \mathcal{S} is an essential sequence of partitions for f .

Hence, for every k fixed, the function

$$h_k(\mu) = h_\mu(\alpha_k),$$

is the infimum of continuous functions over $\mathcal{M}(f)$, and then is upper semicontinuous. Here $h_\mu(\alpha_k)$ is the entropy of the partition α_k for f . The following proposition gives us a very useful way to prove non existence of symbolic extensions. It was also proved in [16].

3.2. Proposition. *Let $f \in \mathcal{R}_\mathcal{S}$ and suppose \mathcal{E} be some compact subset in $\mathcal{M}(f)$ such that there exists a positive real number ρ_0 such that each $\mu \in \mathcal{E}$ and $k > 0$,*

$$\limsup_{v \in \mathcal{E}, v \rightarrow \mu} [h_v(f) - h_k(v)] > \rho_0.$$

Then,

$$h_{\text{sex}}(f) = \infty.$$

Recall $H_n(f)$ denotes the set of hyperbolic periodic points with period smaller or equal than n , and let $H(f) = \cup_n H_n(f)$. By Pugh's closing lemma the set of diffeomorphisms \mathcal{R}_1 formed by f with $H(f) \neq \emptyset$ is open and dense in $\text{Diff}_\omega^1(M)$. Hence, it's well defined $\tau(f)$ as the smallest period of the elements in $H(f)$ for every $f \in \mathcal{R}_1$, and then let $\mathcal{R}_{1,m} \subset \mathcal{R}_1$ be the set of diffeomorphisms f with $\tau(f) = m$. Note, \mathcal{R}_1 is a disjoint union of $\mathcal{R}_{1,m}$.

Now, for each $f \in \mathcal{R}_1$ we define

$$\chi(f) = \sup\{\chi(p, f) : p \in H(f) \text{ and } \tau(p, f) = \tau(f)\}.$$

Then, $\chi(f) > 0$ and depends continuously on $f \in \mathcal{R}_1$.

Recalling that $\mathcal{A} \subset \text{Diff}_\omega^1(M)$ is the set of Anosov diffeomorphisms, let $\mathcal{R}_{2,m} = \mathcal{R}_{1,m} \setminus \text{cl}(\mathcal{A})$, which implies $\mathcal{R}_1 \setminus \text{cl}(\mathcal{A}) = \bigcup_m \mathcal{R}_{2,m}$.

Suppose now that Λ is an f -invariant periodic set with basis Λ_1 and $\alpha = A_1, A_2, \dots, A_s$ some finite partition of M . We say that Λ is *subordinate* to α if for each positive integer n , there exists an element $A_{i_n} \in \alpha$ such that $f^n(\Lambda_1) \subset A_{i_n}$. Hence, if $\mu \in \mathcal{M}(f|\Lambda)$ then $h_\mu(\alpha) = 0$.

Now, given a positive integer n , we say that a diffeomorphism f satisfies property \mathcal{S}_n if for every $p \in H_n(f)$ with $\chi(p, f) > \frac{\chi(f)}{2}$,

1. There exists a hyperbolic basic set of zero dimension $\Lambda(p, n)$ for f such that

$$\Lambda(p, n) \cap \partial\alpha_n = \emptyset \text{ and } \Lambda(p, n) \text{ is subordinate to } \alpha_n.$$

3. There exists an ergodic measure $\mu \in \mathcal{M}(\Lambda(p, n))$ such that

$$h_\mu(f) > \chi(p, f) - \frac{1}{n}.$$

4. For every ergodic measure $\mu \in \mathcal{M}(\Lambda(p, n))$, we have

$$\rho(\mu, \mu_p) < \frac{1}{n}.$$

5. For every periodic point $q \in \Lambda(p, n)$, we have

$$\chi(q, f) > \chi(p, f) - \frac{1}{n}.$$

Given positive integers $m \leq n$, let $\mathcal{D}_{m,n} \subset \mathcal{R}_{2,m}$ be the subset of diffeomorphisms f satisfying property \mathcal{S}_n .

Since periodic points in $H_n(f)$ with smallest positive lyapunov exponent bigger than $\chi(f)/2$ are finite, directly from Proposition 3.1, conditions (3), (4) and (5) above are satisfied for diffeomorphisms in an open and dense subset of $\mathcal{R}_{2,m}$. Now, fixed some partition α_n we can take a smaller open set U where we build the hyperbolic set $\Lambda(p, n)$, as we can see in the proof of Proposition 3.1 in the next section, in order to obtain that the set $\Lambda(p, n)$ is subordinate to α_n . Therefore, since this is a robust property we have proved the following lemma.

3.3. Lemma. *For positive integers $m \leq n$, $\mathcal{D}_{m,n}$ is open and dense in $\mathcal{R}_{2,m}$.*

Now, using property \mathcal{S}_n and the above lemma the proof of Theorem D is similar to the proof of Theorem 1.3 in [16], but for convenience we reproduce it again.

Proof of Theorem D: Let

$$\mathcal{R}_2 = \bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{D}_{m,n}.$$

By Lemma 3.3, we have that $\mathcal{R} = \mathcal{R}_S \cap (\mathcal{R}_2 \cup \mathcal{A})$ is a residual set in $\text{Diff}_\omega^1(M)$.

What we show now is that all non Anosov diffeomorphism $f \in \mathcal{R}$ has no symbolic extension, i.e., $h_{\text{sex}}(f) = \infty$.

Let $f \in \mathcal{R}$ be a non Anosov diffeomorphism. Now, we define

$$\mathcal{E}_1 = \left\{ \mu_p; \text{ such that } p \in H(f) \text{ and } \chi(p, f) > \frac{\chi(f)}{2} \right\},$$

and let \mathcal{E} denotes its closure in $\mathcal{M}(f)$.

Using this set and property \mathcal{S}_n , we show that the hypothesis of Proposition 3.2 are satisfied if we take $\rho_0 = \frac{\chi(f)}{2}$. For this, it's enough to verify the hypothesis for every $\mu_p \in \mathcal{E}_1$, and fixed $k \in \mathbb{N}$.

Hence, given $n \in \mathbb{N}$ big enough, since $f \in \mathcal{R}$, there exists a hyperbolic basic set $\Lambda(p, n)$ for f subordinate to α_n , and an ergodic measure $\nu_n \in \mathcal{M}(\Lambda(p, n))$ such that $\rho(\nu_n, \mu_p) < 1/n$ and

$$(2) \quad h_{\nu_n}(f) > \chi(p, f) - \frac{1}{n}.$$

Since we can suppose $n > k$, we have that α_n is smaller than α_k and then $\Lambda(p, n)$ is also subordinate to α_k . Hence, as $\nu_n \in \mathcal{M}(\Lambda(p, n))$

$$(3) \quad h_k(\nu_n) = 0.$$

Therefore, we have that $\nu_n \rightarrow \mu_p$, when $n \rightarrow \infty$, and

$$|h_{\nu_n}(f) - h_k(\nu_n)| = h_{\nu_n}(f) > \chi(p, f) - \frac{1}{n} > \rho_0,$$

where the last inequality is satisfied for big values of n , since $\mu_p \in \mathcal{E}_1$.

To complete the proof we need to show that ν_n is in \mathcal{E} , for every n . To see this, we use that ν_n is approximated by periodic measures since is an ergodic measure supported in a hyperbolic basic set. That is, there exist $q_{m,n} \in \Lambda(p, n)$, hyperbolic periodic points of f , such that $\mu_{q_{m,n}}$ converges to ν_n in the weak topology. This way, our work is reduced to show that $\mu_{q_{m,n}} \in \mathcal{E}_1$, which is direct from item 5 of property \mathcal{S}_n , provided $f \in \mathcal{R}$. The proof of theorem D is complete. \square

4. SYMPLECTIC PERTURBATIONS: PROOF OF PROPOSITION 3.1

Before going into the proof of the proposition let us recall some basic fact about symplectic structure. Let (V, ω) be a symplectic vector space of dimension $2n$. For any subspace $W \subset V$ its *symplectic orthogonal* is defined as

$$W^\omega = \{v \in V; \omega(v, w) = 0 \text{ for all } w \in W\}.$$

The subspace W is called *symplectic* if $W^\omega \cap W = \{0\}$. W is called *isotropic* if $W \subset W^\omega$, that is $\omega|_{W \times W} = 0$. A special case of isotropic subspace is a *Lagrangian* subspace, i.e., when $W = W^\omega$. For a symplectic manifold (M, ω) and a symplectic diffeomorphism f it is easy to see that for any point on an unstable (stable) manifold of a hyperbolic periodic point, the tangent space to unstable (stable) manifold is a Lagrangian subspace.

The proof of main proposition is done in three steps where the second and third are the main ones and use the symplectic structures.

Let $f \in \mathcal{H}$ be a non Anosov diffeomorphisms.

- Step 1- We find g_1 C^1 -close to f such that p is still a hyperbolic periodic point of g_1 , and g_1 exhibits one homoclinic tangency between $W^s(o(p), g_1)$ and $W^u(o(p), g_1)$. Moreover, $g_1 = Df_p$ in a small neighborhood of the orbit of p (in local symplectic coordinates).
- Step 2- We find g_2 C^1 -close to g_1 where g_2 admits a segment of line of homoclinic tangency. We should perform perturbations in the symplectic high dimensional setting.
- Step 3- Finally, we perturb g_2 to obtain g with a hyperbolic invariant set satisfying the properties required by the proposition. All C^1 -perturbations of g also share the same property for the corresponding hyperbolic set.

Proof of step 1: The way, to create homoclinic tangency is in the lines of the proof of Newhouse (step 6, Theorem 1.1 in [18]). After that, we use a pasting lemma of Arbieto-Matheus [3] and continuity of compact parts of stable and unstable manifolds to obtain a tangency and linearization in a neighbourhood of the periodic point. We should point out that because of high dimensions of stable and unstable manifolds, by homoclinic tangency we obtain at least one (it can be unique) common direction between the tangent spaces of these manifolds at the point of tangency.

Proof of step 2: For simplicity we suppose p is a hyperbolic fixed point of g_1 , and let V be a neighborhood of p where in local symplectic coordinates g_1 is linear, with $E_p^s = \mathbb{R}^n \times \{0\}^n$ and $E_p^u = \{0\}^n \times \mathbb{R}^n$. Moreover, by Darboux's Theorem, we can also suppose in V that ω is the standard 2-form for \mathbb{R}^{2n} , $\omega = \sum dx_i \wedge dy_i$.

Let q be the point of homoclinic tangency between $W_{loc}^s(p, g_1)$ and $W^u(p, g_1)$, such that $q \in V$ and $g_1^{-1}(q) \notin V$. Hence, we can take one small neighborhood $U \subset V$ of q such that $g_1^{-1}(U) \cap V = \emptyset$. We denote by D the connected component of $W^u(p, g_1) \cap U$ that contains q .

We want now to perturb g_1 in order to get an interval of homoclinic tangency. Since stable (unstable) manifolds is a graphic, it's not difficult to do this in the conservative scenario using the point of tangency q . In symplectic case this may be done using the fact that stable (unstable) manifold is a lagrangian manifold as we explain it below.

First, we consider another symplectic coordinate on U in order to simplify the notation such that q is the origem, and we have the following

$$W_{loc}^s(p, g_1) \cap U = \{y_1 = y_2 = \dots = y_n = 0\} \cap U,$$

$$T_q D = \{y_1 = x_2 = \dots = x_n = 0\},$$

and so

$$W_{loc}^s(p, g_1) \cap U \cap T_q D = \{e_1\},$$

where we are considering $\{e_1, \dots, e_n, \dots, e_{2n}\}$ as the canonical basis of \mathbb{R}^{2n} . Note we are using that $\dim(T_q W^s(p, g_1) + T_q W^u(p, g_1)) = 2n - 1$, which we can suppose after some perturbation if necessary.

The following lemma is the technical point that allows us to build the interval of homoclinic tangency for symplectomorphisms.

4.1. Lemma. *There exists a symplectic diffeomorphism $\phi : U \rightarrow \mathbb{R}^{2n}$ over its image, C^1 close to identity map Id in a small neighborhood of q , such that $\phi(D) \cap W_{loc}^s(p, g_1) \cap U$ contains one segment of line.*

Proof. Just here we use coordinates (x, y) with respect to the following decomposition of the space $\mathbb{R}^{2n} = E \oplus F$, where E and F are generated by $\{e_1, e_{n+2}, \dots, e_{2n}\}$ and $\{e_2, \dots, e_{n+1}\}$, respectively. Recall that $E = T_q D$, and $q = (0, 0)$ by the choice of the coordinate.

Now, since D is locally a graphic of one function with the same class of differentiability that g_1 , there exists a C^1 map $j : B \subset \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$, $j(x) = (x, r(x))$, such that $j(B) = D$. Moreover, j is such that $Dr(0) = 0$, and since $D \subset W^u(p, g_1)$ is a lagrangian submanifold, we have $j^* \omega = 0$, where $j^* \omega$ is the pull-back of the form ω by j . Analogously, if $i : \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ is the natural inclusion, $i(x) = (x, 0)$, we have $i^* \omega = 0$ (recall ω in U is the standard 2-form on \mathbb{R}^{2n}).

Let us define $\phi : U \rightarrow \mathbb{R}^{2n}$ by $\phi(x, y) = (x, y - r(x))$. Taking U smaller, if necessary, ϕ is in fact a diffeomorphism from U into its image and C^1 near Id , since $Dr(0) = 0$. Hence, to conclude the lemma we need to show that ϕ is indeed symplectic. Denoting the projection in the first coordinate by $\pi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$, $\pi(x, y) = x$, we can rewrite ϕ in the following way $\phi = Id + i \circ \pi - j \circ \pi$. Then,

$$\phi^* \omega = \omega + \pi^* i^* \omega - \pi^* j^* \omega = \omega,$$

where we use that $i^* \omega = j^* \omega = 0$ in the second equality. Therefore, the lemma is proved. \square

Using the pasting lemma of Arbieto-Matheus [3] in symplectic case and the map ϕ given by Lemma 4.1 we can find $R : U \rightarrow U$ C^1 close to identity Id , with $R = \phi$ in a small neighborhood of q , and $R = Id$ outside another small neighborhood containing the last one. Hence, considering $\tilde{R} : M \rightarrow M$ with $\tilde{R} = Id$ in U^c and $\tilde{R} = R$ in U , and taking $g_2 = \tilde{R} \circ g_1$ we have a C^1 perturbation of g_1 that coincides with g_1 in $(g_1^{-1}(U))^c$. Moreover, the most important is that this perturbation exhibits an interval of homoclinic tangency as we wanted. More precisely, there is one segment of line $I \subset W_{loc}^s(p, g_2) \cap W^u(p, g_2) \cap U$. Note that I is in the space generated by the unit vector e_1 , and after some symplectic coordinate changed inside U , we can suppose $I \subset \{(x_1, 0, \dots, 0), -2a \leq x_1 \leq 2a\}$, for some $a > 0$ small enough and usual coordinates of \mathbb{R}^{2n} .

Proof of Step 3: The idea now is to use this interval of tangency to create hyperbolic sets with the properties required by the proposition. Let N be a big positive integer and $\delta > 0$ an arbitrary small real number. As before (using pasting lemma) we can find a symplectic diffeomorphism $\Theta : M \rightarrow M$, $\delta - C^1$

near Id , $\Theta = Id$ in U^c and

$$\Theta(x, y) = \left(x_1, \dots, x_n, y_1 + A \cos \frac{\pi x_1 N}{2a}, y_2, \dots, y_n \right), \text{ for } (x, y) \in B(0, r) \subset U,$$

for $A = \frac{2Ka\delta}{\pi N}$ and $r > 0$ small enough, where K is some constant depending only on the symplectic coordinate on U . Hence, $g = \Theta \circ g_2$ is $\delta - C^1$ close to g_2 and moreover $g = g_2$ in the complement of $g_2^{-1}(U)$. Although the diffeomorphism g depends on N , we always denote these diffeomorphisms by g . We would like to note this perturbation is an adaptation of Newhouse's snake perturbation for higher dimensions, i.e., it destroys the interval of tangency and creates N transversal homoclinic points for p inside U .

Using the function Θ we choose two good points on unstable manifold of p for g , $z_1 = \Theta(-a, 0, \dots, 0)$ and $z_2 = \Theta(a, 0, \dots, 0)$. Now, we consider γ_1 and γ_2 two transversal disks to unstable manifold $W^u(p, g)$ at z_1 and z_2 , respectively.

From now on we use the symplectic coordinate on V fixed before. Note that g is equal to g_1 inside V and so g is linear in V .

Given a set E , we denote by $C(E, x)$ the connected component of E containing x . By λ -Lema and choice of γ_1 and γ_2 , $C(g^{-j}(\gamma_1) \cap V, g^{-j}(z_1))$ and $C(g^{-j}(\gamma_2) \cap V, g^{-j}(z_2))$ accumulate on $W_{loc}^s(p, g)$ for big values of $j > 0$.

Hence, if $D^s = W_{loc}^s(p, g) \cap U$ then for j big enough we can define the rectangle $D_j = D^s \times D_j^u$ as being the cartesian product between D^s and D_j^u , where D_j^u is the smallest possible disk in $\{(0, \dots, 0, y_1, \dots, y_n), y_i \in \mathbb{R}\}$ such that $\pi_2(C(g^{-j}(\gamma_i) \cap V, g^{-j}(z_i))) \subset D_j^u$, for $i = 1, 2$. Here $\pi_2(x, y) = y$ stands for the projection on the second n th-coordinates of \mathbb{R}^{2n} , and recall we are considering V inside Euclidean space with $E_p^s = \mathbb{R}^n \times \{0\}^n$ and $E_p^u = \{0\}^n \times \mathbb{R}^n$.

Let $J \subset U$ be some small enough disk inside the unstable manifold $W^u(p, g)$ containing the N transversal homoclinic points built before, and let $T \gg 0$ such that $g^{-T}(J) \subset V$, and moreover $g^{-T}(\gamma_i)$, $i = 1, 2$, is close to $W_{loc}^s(p, g)$. We denote by $\tilde{\Gamma}$ the $A/2$ -neighborhood of J , and define $\Gamma = g^{-T}(\tilde{\Gamma})$.

Now, let t_0 be the smallest positive integer such that $C(g^{-t_0}(\gamma_i), g^{-t_0}(z_i))$ is $A/2 - C^1$ close to $W_{loc}^s(p, g)$, $i = 1, 2$. Note that if $t' \geq t_0$, and $g^{t'-T}(D_{t'}) \subset \Gamma$, then $g^{t'}(D_{t'}) \cap (D_{t'})$ contains N disjoint connected components. Hence, we consider $z_3 = (b, 0, \dots, 0)$ and $z_4 = (b', 0, \dots, 0)$ two points on local stable manifold of p , where b and b' are the left and right boundary points in the first coordinate of $W_{loc}^s(p, g) \cap U$. Also, let γ_3 and γ_4 be two transversal disks to $W_{loc}^s(p, g)$ at z_3 and z_4 , respectively. By λ -lemma again we can define t_1 as the smallest possible positive integer such that

$$C(g^{t_1}(\gamma_i), g^{t_1}(z_i)) \cap C(g^{-T}(\gamma_j), g^{-T}(z_j)) \cap \Gamma \neq \emptyset, \text{ for } j = 1, 2 \text{ and } i = 3, 4.$$

Finally, we define $t = \max\{t_0, t_1 + T\}$, see figure 1. Note t depends on N since t_0 and t_1 depends, and also observe that t goes to infinity when N goes.

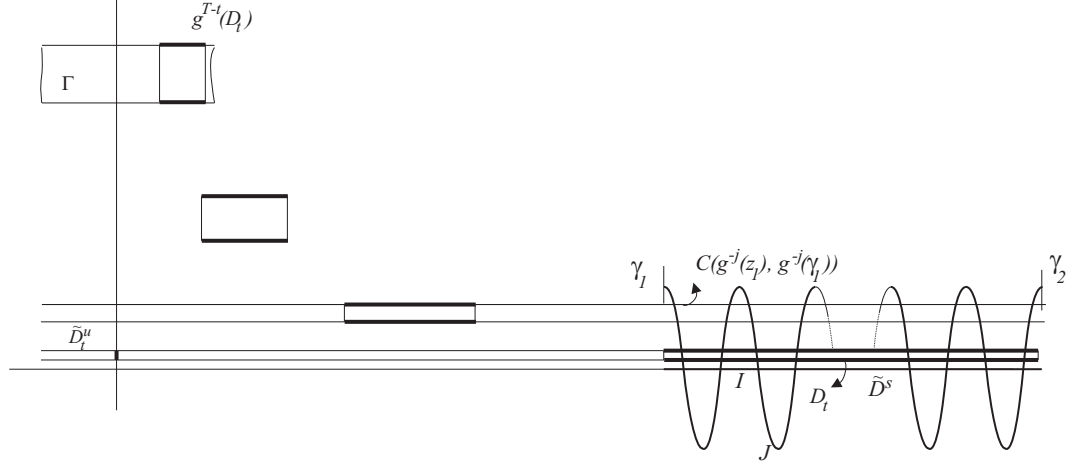


FIGURE 1.

By the comments above and choice of t , we have that $g^t(D_t) \cap D_t$ has N disjoint connected components, and t is the smallest possible one such that D_t is $A/2 - C^1$ close to $W_{loc}^s(p, g)$ and $g^t(D_t)$ is $A/2 - C^1$ close to $J \subset W^u(p, g)$. Therefore, since we have a horseshoe with N legs, the maximal invariant set in D_t for g^t

$$\tilde{\Lambda}(p, N) = \bigcap_{j \in \mathbb{Z}} g^{tj}(D_t)$$

is a hyperbolic set with dynamics conjugated to a shift of N symbols. Then $h(g^t|_{\tilde{\Lambda}(p, N)}) = \log N$, and taking

$$\Lambda(p, N) = \bigcup_{j=1}^t g^j(\tilde{\Lambda}(p, N))$$

we have $h(g|_{\Lambda(p, N)}) = \frac{1}{t} \log N$.

The following lemma is the main point in this step.

4.2. Lemma. *For A and t defined as before, there exists a positive integer K_1 independent of A , such that*

$$A < K_1 \min\{\|Dg_p^{-t}|E^u\|, \|Dg_p^t|E^s\|\}.$$

Proof. Since V is a neighborhood of p where g is linear, if m is the biggest one such that $g^j(x) \in V$ for $0 \leq j \leq m$, there exist constants K_2 and K_3 depending on the symplectic coordinate on V such that

$$(4) \quad K_2 \|Dg_p^m|E^u\|^{-1} \leq d(x, W_{loc}^s(p, g)) \leq K_3 \|Dg_p^{-m}|E^u\|,$$

for $x \in V$. Analogously, if m is the biggest one such that $g^{-j}(x) \in V$ for $0 \leq j \leq m$, then there exist constants K_4 and K_5 such that

$$(5) \quad K_4 \|Dg_p^{-m}|E^s\|^{-1} \leq d(x, W_{loc}^u(p, g)) \leq K_5 \|Dg_p^m|E^s\|.$$

Now, by choice of t , either there exists $z \in D_t$ such that

$$(6) \quad d(g(z), W_{loc}^s(p, g)) \geq A/2,$$

or there exists $z \in g^t(D_t)$ such that

$$(7) \quad d(g^{-1}(z), J) \geq A/2.$$

Suppose the first case. Recall that for $j > T$ the rectangle D_j is defined and moreover $D_j \subset V$, which implies $g(z), g(g(z)), \dots, g^{t-1-T}(g(z)) \in V$. Hence, using inequality (4) we have

$$\frac{A}{2} \leq K_3 \|Dg_p^{-t+T+1}|E^u\|.$$

On the other hand, using inequality (5) and the neighborhood Γ , we can do the same thing for the second case, obtaining

$$\frac{A}{2} \leq K_5 \|Dg_p^{t-1-T}|E^s\|.$$

And then, since Dg is bounded and T is independent of A we can find K_1 as we claimed. □

From now on fix n a large positive integer as Proposition 3.1 requires.

Since $A = \frac{2Ka\delta}{\pi N}$, using Lemma 4.2 and for N big enough, we have

$$\frac{1}{t} \log N > \min \left\{ \frac{1}{t} \log \|Dg_p^{-t}|E^u\|^{-1}, \frac{1}{t} \log \|Dg_p^t|E^s\|^{-1} \right\} - \frac{1}{2n}.$$

Observe now, when t goes to infinity the above minimum converges to the minimum between the smallest positive Lyapunov exponent, $\chi(p, g)$ as defined before, and the absolute value of the biggest negative Lyapunov exponent of p for g . Moreover, since we are in the symplectic scenario these two numbers are equal. Therefore, provided t goes to infinity when N goes, we can find a positive integer N_1 such that

$$\frac{1}{t} \log N_1 > \chi(p, g) - \frac{1}{n}.$$

Which implies that it's possible to find some C^1 -perturbation g of f such that

$$h(g|\Lambda(p, N_1)) > \chi(p, g) - \frac{1}{n}.$$

For general case, when p is not a fixed point of g_1 , i.e., $\tau(p, g_1) > 1$, we have that $q \in W_{loc}^s(p, g_1) \cap W^u(f^j(p), g_1)$, for some $0 \leq j < \tau(p, g_1)$. Then as we did before, we can find some perturbation g of g_1 and $t = \tau(p, g)\tilde{t} + j$ such that g^t has a hyperbolic basic set $\tilde{\Lambda}(p, N)$. Moreover, there is a relation between the norm of $Dg^{\tau(p, g)}$ and A as in the Lemma 4.2, changing t by \tilde{t} . Hence, we can find N_1 such that

$$(8) \quad h(g|\Lambda(p, N)) > \chi(p, g) - \frac{1}{n}, \text{ for } N \geq N_1.$$

Now, since $\Lambda(p, N)$ is conjugated to the product between some finite permutation dynamics and the shift of N symbols, there exists an ergodic measure $\mu_N \in \mathcal{M}(\Lambda(p, N))$ that maximizes the topological entropy. Hence, directly from (8)

$$(9) \quad h_{\mu_N}(g) > \chi(p, g) - \frac{1}{n}, \text{ for } N \geq N_1.$$

We suppose from now on that p is fixed, being the general case similar deduced as we did before. Next, we find a positive integer N_2 such that if $\mu \in \mathcal{M}(f|\Lambda(g, N_2))$ is ergodic then $\rho(\mu, \mu_p) < 1/n$ as required. For this, given $\zeta > 0$ arbitrary small it's enough to find $N = N(\zeta)$ such that (orbit of) any point of $\Lambda(p, N)$ visits very frequently the ball of radius ζ and center p .

Provided p is a hyperbolic fixed point we have

$$\bigcap_{i \in \mathbb{Z}} g^i(V) = \{p\}.$$

Hence, given $\zeta > 0$ arbitrary small, there exists a positive integer $n_1 \geq T$, depending on ζ , such that for every $n_2 \geq n_1$

$$\text{diam} \bigcap_{-n_2 \leq i \leq n_2} g^i(V) < \zeta.$$

Now, if $\overline{V} = \bigcap_{i=0}^{l n_1} g^{-i}(V)$ and $z \in \overline{V}$, then for every $r \in [n_1, (l-1)n_1]$ we have that

$$g^r(z) \in \bigcap_{|i| < n_1} g^i(V) \subset B_\zeta(p).$$

So, the fraction of time in $[0, l n_1)$ that the orbit of z stay in $B_\zeta(p)$ is $\frac{l-2}{l}$.

Recall that t is the period of the periodic set $\Lambda(p, N)$ of g , and let us define $k = t - T$. Given N big enough, let $l \in \mathbb{N}$ be such that $(l+1)n_1 \geq k > l n_1$. Since for every $z \in \Lambda(p, N)$ there exists $r \in [0, t)$ such that $g^r(z) \in \overline{V}$, the frequency of the orbit of z passing in $B_\zeta(p)$ is bigger than

$$\frac{(l-2)n_1}{(l+1)n_1 + T}.$$

Provided $l \rightarrow \infty$ when $N \rightarrow \infty$, given $\zeta_1 > 0$ we can choose N_2 such that the frequency of the orbit of every $z \in \Lambda(p, N)$ passing in $B_{\zeta}(p)$ is bigger than $1 - \zeta_1$, and then choosing ζ_1 smaller if necessary we have

$$(10) \quad d(\mu, \mu_p) < \frac{1}{n}, \text{ for every ergodic measure } \mu \in \mathcal{M}(\Lambda(p, N)), N \geq N_2.$$

Finally, we find N_3 in order to obtain property (d) for $\Lambda(p, N)$, $N \geq N_3$. We define

$$V_k^u = V \cap g(V) \cap \dots \cap g^k(V), \text{ and} \\ V_k^s = V \cap g^{-1}(V) \cap \dots \cap g^{-k}(V).$$

Given vectors $v, w \in \mathbb{R}^{2n}$ and subspaces $E, F \subset \mathbb{R}^{2n}$ we define

$$\text{ang}(v, w) = \left| \tan \left[\arccos \left(\frac{\langle v, w \rangle}{\|v\| \|w\|} \right) \right] \right|,$$

$$\text{ang}(v, E) = \min_{w \in E, \|w\|=1} \text{ang}(v, w) \quad \text{and} \quad \text{ang}(E, F) = \min_{w \in E, \|w\|=1} \text{ang}(w, F).$$

4.3. Remark. Another definition of the angle between two subspaces in literature is the following: If $\mathbb{R}^n = E \oplus F$ is some decomposition, let $L : E^\perp \rightarrow E$ be the linear map such that $F = \{w + Lw; w \in E^\perp\}$, and then some authors define the angle between E and F as $\|L\|^{-1}$. Nevertheless, there is an equivalence between this definition and the one presented here.

We need the following lemma.

4.4. Lemma. With above definitions, there exists constant K_6 such that if $z \in V_{k'}^s$, $v \in \mathbb{R}^{2n} \setminus E_p^s$ and $\text{ang}(g^k(v), E_p^s) \geq 1$, then

$$|Dg^k(z)(v)| \geq K_6 \|Dg_p^{-k}\|^{-1} |v| \min\{\text{ang}(v, E_p^s), 1\}.$$

Proof. Using the decomposition of \mathbb{R}^{2n} fixed on V , we have $v = (v^s, v^u)$, $v^{s(u)} \in E_p^{s(u)}$, for every $v \in \mathbb{R}^{2n}$. Let $|v|' = \max\{|v^s|, |v^u|\}$ be the maximum norm.

Since $E_p^{s\perp} = E_p^u$, and $Dg^k(z) = Dg_p^k$ if $z \in V$, we have

$$(11) \quad \text{ang}(v, E_p^s) = \frac{|v^u|}{|v^s|} \quad \text{and} \quad 1 \leq \text{ang}(Dg^k(z)(v), E_p^s) = \frac{|Dg_p^k(v^u)|}{|Dg_p^k(v^s)|}.$$

Then,

$$\begin{aligned} |Dg^k(z)(v)|' &= |Dg_p^k(v^u)| \\ &\geq \|Dg^{-k}|E_p^u|\|^{-1} |v^u|, \\ &= \|Dg_p^{-k}|E_p^u|\|^{-1} |v^s| \text{ang}(v, E^s); \end{aligned}$$

which implies

$$(12) \quad |Dg^k(z)(v)|' \geq \|Dg_p^{-k}|E^u|^{-1}|v|' \min\{\text{ang}(v, E_p^s), 1\}.$$

Therefore, by the equivalence between norms, the result follows. \square

Recall now,

$$\Lambda(p, N) = \bigcup_{i=0}^{t-1} g^i(\tilde{\Lambda}(p, N))$$

is a hyperbolic set for g , with $\tilde{\Lambda}(p, N) \subset V$, and $t = k + T$ where $g^i(\tilde{\Lambda}(p, N)) \subset V$ for $0 \leq i \leq k$. Moreover, by construction of $\tilde{\Lambda}(p, N)$ we know that the hyperbolic decomposition $T_{\Lambda(p, N)}M = \tilde{E}^s \oplus \tilde{E}^u$ is such that $\tilde{E}^s(z)$ and $\tilde{E}^u(g^k(z))$ are close to $E^s(p)$ and $E^u(p)$, respectively, for every $z \in \tilde{\Lambda}(p, N)$. In particular, $\text{ang}(Dg^k(z)(v), E^s(p)) > 1$ for $v \in \tilde{E}^u(z)$.

Hence, we can use Lemma 4.4 to find a constant K_6 , such that for every $z \in \tilde{\Lambda}(p, N)$ and $v \in \tilde{E}^u(z)$,

$$(13) \quad |Dg^r(z)(v)| \geq (CK_6)^l \|Dg_p^{-k}|^{-l}|v|, \text{ for } r = l(k + T), \quad l \in \mathbb{N}.$$

where

$$C = \inf_{z \in V \setminus g^{-1}(V), |v|=1} \|Dg^T(z)(v)\|.$$

Therefore, it's not difficult to see that for N big enough, all points in $\tilde{\Lambda}(p, N)$ have positive lyapunov exponents bigger than $\chi(p, g) - 1/n$. In particular, we can choose N_3 in order to get $k \gg T$, such that for any periodic point $q \in \Lambda(p, N)$, $N > N_3$,

$$\chi(q, g) \geq \chi(p, g) - \frac{1}{n}.$$

Hence, if we take $\Lambda(p, n) = \Lambda(p, N)$ for $N = \max\{N_1, N_2, N_3\}$, the properties of proposition are satisfied for the perturbation g of f .

Now, by robustness of the hyperbolic periodic point p and the set $\Lambda(p, n)$, properties (1) and (2) are also satisfied for diffeomorphisms close to g . Recall that μ_n is the one that maximize topological entropy, then in order to prove properties (3) and (4) we just concerned with some neighborhood of the set $\Lambda(p, n)$, and so, the same could be done for diffeomorphisms near g . Which concludes the proof of proposition. \square

5. EXAMPLE OF DISCONTINUITY POINTS FOR TOPOLOGICAL ENTROPY

In order to prove Theorem C, we construct an example of a C^∞ area preserving diffeomorphism over S^2 that is a non upper semi-continuity point for topological entropy in the space $\text{Diff}_\omega^1(S^2)$.

Firstly, let S be any surface different of \mathbb{T}^2 . As S does not accept Anosov diffeomorphism, by Theorem A we have that for generic volume preserving diffeomorphisms of S

$$h(f) = s(f).$$

Hence, using that $s(\cdot)$ is a lower semi-continuous function, if we find a diffeomorphism $f \in \text{Diff}_\omega^1(S^2)$ such that $h(f) < s(f)$, then this is an example where topological entropy is not upper semi-continuous.

In order to find such diffeomorphism, we use the following result of Lai-Sang Young [23].

5.1. Theorem. *Let $\phi : \mathbb{R} \times M \rightarrow M$ be a flow in a 2-dimensional manifold M . Then, the diffeomorphism $\phi_t = \phi(t, \cdot)$ over M has zero topological entropy, i.e, $h(\phi_t) = 0$, for every t .*

By this result and the above discussion, to prove Theorem C it's enough to find a hamiltonian flow in S^2 with a hyperbolic periodic orbit.

In what follows we describe the construction of such example which uses the well known mathematical pendulum, see [19]. Recall that a vector field X_H over a compact symplectic manifold (M, ω) is Hamiltonian iff there exists a smooth map $H : M \rightarrow \mathbb{R}$ such that

$$\omega(X_H, \cdot) = dH.$$

Also, recall $\phi_t = \phi(t, \cdot) : M \rightarrow M$ is a symplectic diffeomorphism, for every $t \in \mathbb{R}$, where ϕ is the flow generated by X_H . Note that in dimension two the space of conservative diffeomorphisms coincides with the symplectic one.

From now on we consider the symplectic manifold (S^2, ω) the two dimensional sphere, with $\omega(x) = \langle x, u \times v \rangle$, for $x \in S^2$ and $u, v \in T_x S^2$, some induced area form over S^2 . If we give for S^2 cylindrical polar coordinates (θ, z) , $0 \leq \theta < 2\pi$ and $-1 < z < 1$, away from its poles, we can verify that $\omega = d\theta \wedge dz$.

Let $H_1(\theta, z) = z$ be the height function over the sphere, and X_{H_1} be the Hamiltonian vector field generated by H_1 . Note the flow generated by X_{H_1} has no hyperbolic periodic orbits, more precisely the poles are non-hyperbolic singularities, and the flow far from them is $\phi(t, (\theta, z)) = (\theta + t, z)$, i.e., rotations.

On the other hand, we can use the famous mathematical pendulum on $S^1 \times \mathbb{R}$ to build hyperbolic periodic orbits in the previous flow. Let $H_2 : S^1 \times \mathbb{R} \rightarrow \mathbb{R}$ be the total energy of the pendulum, $H_2(\theta, z) = \frac{1}{2}z^2 - \cos \theta$, then the Hamiltonian vector field X_{H_2} on the cylinder gives us the phase portrait of the pendulum. We observe that the flow generated by X_{H_2} has an unstable equilibrium at $p = (\pi, 0)$. Now, considering $\beta : (-1, 1) \rightarrow \mathbb{R}$ the C^∞ bump function such that $\beta(x) = 1$ if $|x| < 1/2$ and $\beta(x) = 0$ if $|x| > 2/3$, we define $H : S^1 \times (-1, 1) \rightarrow \mathbb{R}$ as follows

$$H(\theta, z) = \beta(|z|)H_2(\theta, z) + (1 - \beta(|z|))H_1(\theta, z).$$

Hence, after some coordinate change we can look for this function over S^2 . In fact, what we did was just to carry the pendulum flow to the sphere by changing the height function on some strip. See figure 2. And finally, X_H is a Hamiltonian vector field on S^2 and the flow generated by it has a hyperbolic singularity as we wanted.

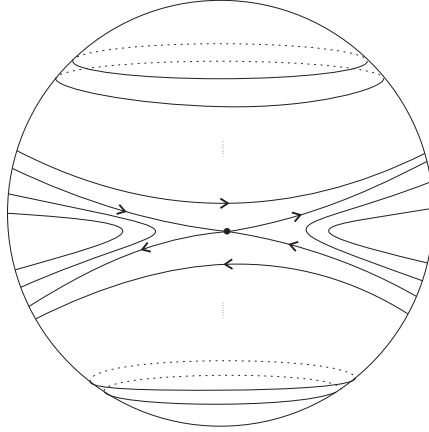


FIGURE 2. Phase portrait of X_H

Acknowledgements: T. C was supported by CAPES-Brazil and FAPESP-Brazil by a Doctoral fellowship. A. T was supported by FAPESP-Brazil and CNPq-Brazil. The authors also wish to point out the excellent research atmosphere at ICMC-USP.

REFERENCES

- [1] F. Abdenur, C. Bonatti and S. Crovisier, Nonuniform Hyperbolicity for C^1 -generic diffeomorphisms, *preprint* (2008) [arXiv:0809.3309](#), To appear in Israel Journal of Mathematics.
- [2] A. Arbieto and T. Catalan, Hyperbolicity in the Volume Preserving Scenario *preprint* (2010) [arXiv:1004.1664](#).
- [3] A. Arbieto and C. Matheus, A pasting lemma and some applications for conservative systems, *Erg. Th. and Dynamic. Sys.*, 27 (2007), 1399-1417.
- [4] M-C. Arnaud, Le "Closing Lemma" en topologie C^1 , *Supplment au Bull. Soc. Math. Fr.*, 74(1998)
- [5] A. Avila, On the regularization of conservative maps, *preprint* (2008) [arXiv:0810.1533](#), To appear in Acta Mathematica.
- [6] R. Bowen, Topological Entropy and Axiom A, *Proc. Symp. Pure Math.*, AMS., Providence RI., 14 (1970), 23-41.
- [7] M Boyle, D. Fiebig and U. Fiebig, Residual entropy, conditional entropy, and subshift covers, *Forum Math.*, 14 (2002), 713-757.
- [8] D. Burguet, C^2 surface diffeomorphisms have symbolic extensions, *preprint* (2010) [arXiv:0912.2018](#).

- [9] J. Buzzi, Intrinsic ergodicity for smooth interval maps, *Israel J. Math.*, 100 (1997), 125-161.
- [10] L. Diaz and T. Fisher, Symbolic extensions for partially hyperbolic diffeomorphisms, *preprint* (2009) [arXiv:0906.2176](#), To appear in *Discrete and Continuous Dynamical Systems*.
- [11] L. Diaz, T. Fisher, M. Pacífico, and J. Vieitez, Entropy-expansiveness for partially hyperbolic diffeomorphisms, *preprint* (2010) [arXiv:1010.0721](#).
- [12] V. Horita and A. Tahzibi, Partial hyperbolicity for symplectic diffeomorphisms, *Ann. I. H. Poincaré* AN 23 (2006), 641-661.
- [13] A. Katok, Lyapunov exponents, entropy and periodic points for diffeomorphisms, *Publications Mathématiques de IHES*, 1980.
- [14] C. Liang, G. Liu and W. Sun, Equivalent Conditions of Dominated Splittings for Volume-Preserving Diffeomorphism, *Acta Math. Sinica* 23 (2007), 1563-1576.
- [15] G. Liao, J. Yang and M. Viana, Entropy of diffeomorphisms away from tangencies, *Private communication*.
- [16] T. Downarowicz and S. E. Newhouse, Symbolic extension and smooth dynamical systems, *Inventiones Mathematicae*, 160 (2005), 453-499.
- [17] S. E. Newhouse, Topological entropy and Hausdorff dimension for area preserving diffeomorphisms of surfaces, *Socit Mathématique de France, Astisque*, 51 (1978), 323-334.
- [18] S. E. Newhouse, Quasi-elliptic periodic points in conservative dynamical systems, *American Journal of Mathematics*, 99, No. 5 (1975), 1061-1087.
- [19] J. Palis and F. Takens, Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations. *Cambridge: Cambridge University Press*, 1993. (Cambridge Studies in Advanced Mathematics).
- [20] M. Rees, A minimal positive entropy homeomorphism of the 2-torus J. London Math. Soc. (2) 23 (1981), no. 3, 537-550
- [21] A. Tahzibi, C^1 Generic Pesin's entropy formula, *C. R. Acad. Sci. Paris, Ser. I* 335 (2002), 1057-1062.
- [22] Z. Xia, Homoclinic points in symplectic and Volume-Preserving diffeomorphisms, *Communications in Mathematical Physics*, 177 (1996), 435-449.
- [23] L. S. Young, Entropy of continuous flows on compact 2-manifolds, *Topology*, 16 (1977), no. 4, 469-471.
- [24] E. Zehnder, Note on smoothing symplectic and volume-preserving diffeomorphisms, *Geometry and topology* (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), pp. 828-854. *Lecture Notes in Math.*, Vol. 597, Springer, Berlin, 1977.

Thiago Catalan
 Instituto de Ciências, Matemática e Computação
 Universidade de São Paulo
 16-33739153 São Carlos-SP, Brazil
 E-mail: catalan@icmc.usp.br

Ali Tahzibi
 Instituto de Ciências, Matemática e Computação
 Universidade de São Paulo
 São Carlos-SP, Brazil
 Email: tahzibi@icmc.usp.br, URL: www.icmc.usp.br/~tahzibi